Chapter 9

VIBRATORY MOTION

There are two areas of interest when discussing oscillatory motion: the mathematical characterization of vibrating structures that generate waves, and the interaction of waves with other waves and with their surroundings. We will examine the former topic in this chapter, the latter in the next chapter.

A.) Vibratory Motion--Basic Concepts:

1.) For any structure to vibrate periodically, there must be a restoring force on the body. A restoring force is a force that is constantly attempting to accelerate the object back toward its *equilibrium position*.

2.) The easiest way to examine vibratory motion is with an example. We will use a spring system:

a.) Consider the mass attached to the spring shown in Figure 9.1. When the spring is neither compressed nor elongated, the mass *feels no force* and is, hence, in a state of equilibrium.

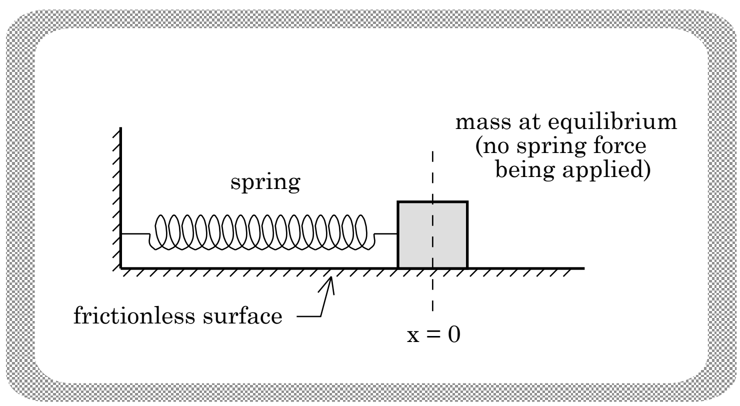


FIGURE 9.1

Note: For this and similar systems, the coordinate axis used to define *mass position* has its origin (i.e., $x = 0$) defined at the body's equilibrium position.

b.) It has been experimentally observed that if an "ideal" spring (i.e., one of those mythical types that loses no energy during oscillation) is displaced a distance Δx (see Figure 9.2), the force F required to displace the spring will be proportional to the displacement Δx. Put another way, if a mass is attached to the spring and the spring is displaced a distance Δx, *the spring will exert a force F on the mass* when released. That force will be proportional to the spring's displacement from its equilibrium position.

Defining *k* as the proportionality constant (units: nt/m), this force is:

$$F = -k\,(\Delta x).$$

Called *Hooke's Law*, this relationship and the motion it describes is called "simple harmonic motion."

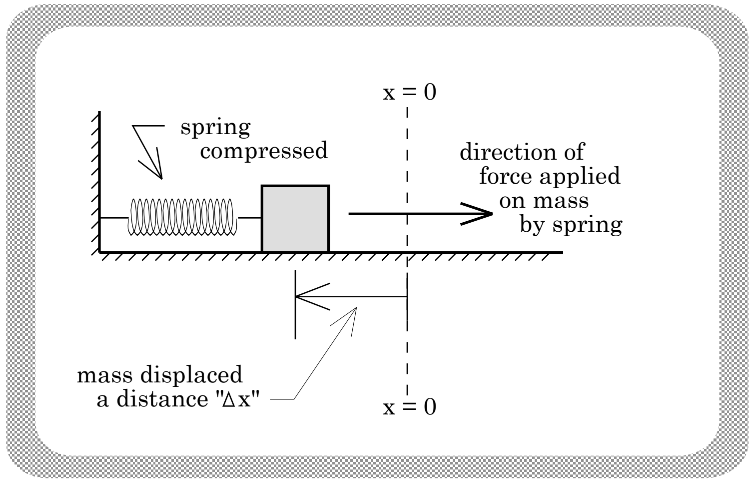


FIGURE 9.2

Note 1: Because Δx is measured *from equilibrium* (i.e., from $x = 0$), a displacement *x-units-long* will equal $\Delta x = x_{final} - x_{initial} = x - 0$. In other words, $\Delta x = x$. As such, HOOKE'S LAW IS ALWAYS WRITTEN:

$$F = -kx.$$

Note 2: Be sure you understand which force Hooke's Law alludes to: it is the force *the spring applies to the mass*, not the force the mass applies to the spring.

Note 3: The negative sign in front of the *kx* term ensures that the force is always directed back toward the equilibrium position. To see this, assume the spring in Figure 9.2 has a spring constant of 2 nt/m and is displaced a distance $x = -.6$ *meters*. The force equation yields:

$$\begin{aligned} F &= -\,[(2 \text{ nt/m})\,(-.6 \text{ m})] \\ &= +1.2 \text{ nts.} \end{aligned}$$

The *direction* of the spring's force on the mass is *positive*, just as common sense would dictate. Without the *negative sign* on the right hand side of the force equation, the mathematics would not accurately model the situation.

3.) Here are some DEFINITIONS needed for the discussion of vibratory systems:

a.) Periodic motion: Any motion that *repeats itself* through time.

b.) Simple harmonic motion: Periodic motion whose force function is of the form $-kx$, where k is a constant and x is the displacement of the structure at some arbitrary point in time.

c.) Frequency (ν): The number of cycles swept through per-unit-time; the MKS units are *cycles per second* (i.e., *hertz*, abbreviated *Hz*). The symbol used for frequency-- ν --is the Greek letter *nu*.

Important Note: *Cycles* is not technically a unit. In many texts, *hertz* is defined as *inverse seconds* (i.e., *1/seconds*). We will use both, depending upon the situation.

d.) Period (T): The time required to sweep through one complete cycle. The units are *seconds per cycle* (or just *seconds*). Note that the *period* and *frequency* of a body's motion are inversely related. That is:

$$T = 1 / \nu.$$

e.) Displacement (x): The distance a vibrating object is from its equilibrium position at a given point in time. *Displacement* is a *time varying* quantity whose units are in *meters* or *centimeters* or whatever the distance units are for the system.

f.) Amplitude (A): The *maximum displacement* x_{max} of an oscillating body. *Assuming the vibratory motion does not lose energy*, the amplitude of the motion remains constant--it does *not* vary with time. *Amplitudes* are measured *from equilibrium* and have the same units as *displacement*.

Note: It is interesting to observe that because the *force function* for a spring is proportional to the spring's displacement ($F = -kx$), the *period* and, hence, *frequency* of a given spring/mass system will be a *constant*. Why?

An oscillation with a *very small displacement* will have a very small distance to travel during one period, but it will also have a *very small spring force* to motivate it. An oscillation with a *very large displacement* will have a very large distance to cover during one period, but it will have a *very large spring force* to help it along. The net result: whether you have big oscillations or small oscillations, it takes the same amount of *time* to oscillate through *one cycle*.

B.) The Mathematics of *Simple Harmonic Motion*:

1.) We would like to derive an expression that defines the *displacement* of a vibrating object from equilibrium as a function of time--i.e., *x(t)*. To generate the appropriate equation, we will examine the vibratory motion of a mass attached to a spring (see Figure 9.3), using Newton's Second Law to evaluate the motion.

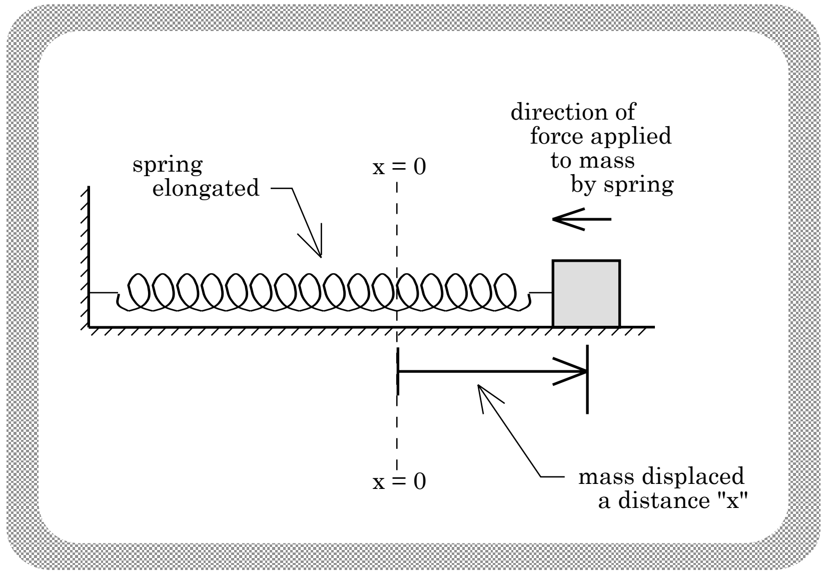


FIGURE 9.3

a.) A *free body diagram* is shown in Figure 9.4. Summing the forces in the horizontal, and leaving the *sign of the acceleration* embedded within the *ma*, we get:

$\underline{\sum F_x}$:

$$-kx = ma$$

$$\Rightarrow \quad a + (k/m)x = 0.$$

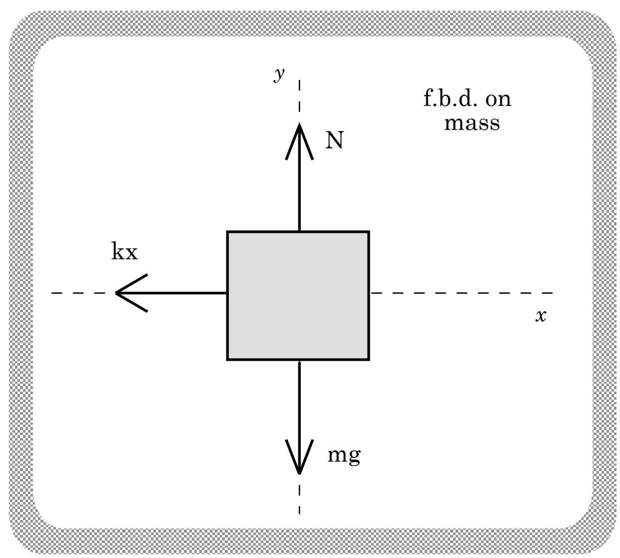


FIGURE 9.4

b.) We know that the acceleration and velocity are related by $a = dv/dt$, and that the velocity and displacement are related by $v = dx/dt$. As such, it is true that $a = d^2x/dt^2$, where the notation used is meant to convey *the second derivative of the position with respect to time.*

c.) Substituting $a = d^2x/dt^2$ into the force expression yields:

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0.$$

d.) What does this equation really say? It suggests that there exists a function *x* such that when you add its *second derivative* to a *constant times itself*, you ALWAYS get zero.

The question is, "What function will do the job?"

The answer is, "A sine wave."

2.) The most general expression for a sine wave (see Figure 9.5a on the next page) is:

$$x(t) = A \sin\left(\omega t + \phi\right),$$

where *A* is the *amplitude* of the displacement (i.e., its maximum possible value); ω is a constant called the *angular frequency* whose units are *radians/second* and whose significance will become clear later; and ϕ is another constant called the *phase shift* whose units are in *radians* and whose significance will also be discussed later.

3.) Using the Calculus on our sine function, we find that if:

$$x(t) = A \sin\left(\omega t + \phi\right).$$

a.) The velocity of the motion will be:

$$\begin{aligned} v(t) &= dx/dt \\ &= \omega A \cos\left(\omega t + \phi\right). \end{aligned}$$

b.) The acceleration of the motion will be:

$$\begin{aligned} a(t) &= dv/dt \\ &= -\omega^2 A \sin\left(\omega t + \phi\right). \end{aligned}$$

c.) The graphs of all three of these functions are found in Figures 9.5a, b, and c.

Note: Notice that the *horizontal axis* is not labeled in time t but rather in $\omega\ t$. Sine waves are functions of angles. Angles must have arguments in angular measure (radians in this case). That means the expression $x = A \sin \boldsymbol{t}$ makes no sense as written--you can't have a sine argument whose units are in *time*. To get around the problem, we modify the time variable by multiplying by ω *radians per second*.

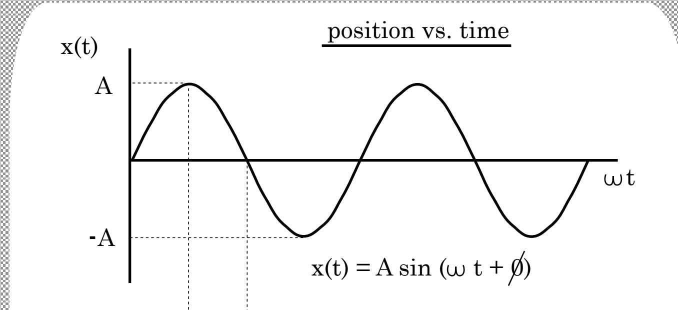


FIGURE 9.5a

d.) The *maximum value* for both a *sine* and a *cosine* function is *1*. This means:

$$v_{max} = \omega A$$

and

$$a_{max} = \omega^2 A$$

(where a_{max} is a magnitude).

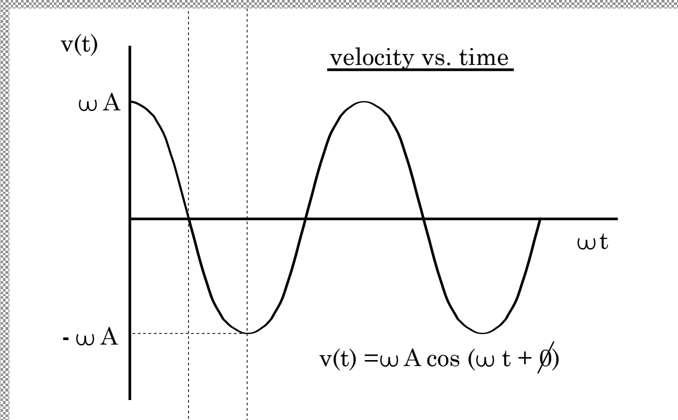


FIGURE 9.5b

4.) Analyzing the graphs:

a.) Look at the *first* long, vertical dotted line spanning Figures 9.5a, b, and c:

i.) The graphs suggest that when the displacement x is maximum-and-positive (i.e., as far to the right of equilibrium as it gets), the acceleration is maximum-and-negative.

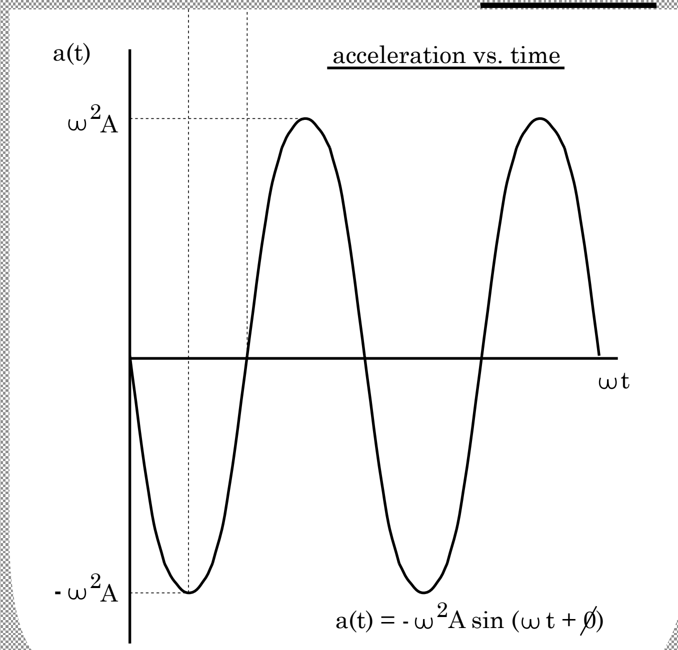


FIGURE 9.5c

Note: A *negative amplitude* value ($-\omega^2 A = -10\ m/s^2$, for instance) does not signify a minimum. The fact that -10 is smaller than +10 on a number line is not relevant here. The value $-\omega^2 A$ is the *largest acceleration possible*; the negative sign simply tells you the *direction* in which it is a maximum.

ii.) Back to the *first vertical line*: When x is at its extremes, the *velocity* of the body is zero. This makes sense. At the extremes the body *stops* before beginning back in the opposite direction.

b.) Look at the *second* long, vertical dotted line spanning Figures 9.5a, b, and c:

i.) The graphs suggest that when x is zero (i.e., the body is at equilibrium), the acceleration is zero. This makes sense. At equilibrium the force applied to the body by the spring is zero, hence zero acceleration would be expected.

ii.) When x is at equilibrium, the *velocity* is a positive or negative <u>maximum</u>, depending upon which direction the body is moving. This also makes sense intuitively. Only when every bit of *acceleration* has been exhausted in motivating the mass back toward the equilibrium point will the velocity be at its maximum. That occurs at equilibrium.

5.) If we go back to Newton's Second Law with this information:

a.) Substituting our sine-related $x(t)$ and $a(t)$ functions back into our force expression (i.e., $a + (k/m)\,x = 0$), we get:

$$[-w^2 A \sin(\omega t + \phi)] + (k/m)\,[A \sin(\omega t + \phi)] = 0.$$

b.) Noting that the *A's* and the *sine functions* cancel, we end up with:

$$-\omega^2 + (k/m) = 0$$
$$\Rightarrow \quad \omega = (k/m)^{1/2}.$$

6.) Evidently, the function $x(t) = A \sin(\omega t + f)$ satisfies the equation $a + (k/m)\,x = 0$ as long as $\omega = (k/m)^{1/2}$.

a.) BIG GENERAL POINT: If you can manipulate a Newton's Second Law equation into the form:

acceleration + (some constant) (displacement) = 0,

you know for certain that:

i.) The motion will be *simple harmonic motion* (versus some other form of oscillatory motion); and

ii.) The *angular frequency* ω of the motion will be equal to the *square root* of the *constant in front of the displacement variable* in the manipulated N.S.L. equation. In the case of a spring, Newton's Second Law yielded:

$$a + (k/m)\,x = 0,$$

and the constant in front of the *position* variable *k/m* was found to be such that:

$$\omega = (k/m)^{1/2}.$$

b.) You might not think much of this revelation now, but it is going to be very useful later when we examine other kinds of vibrating systems.

Before we can look at these *other types of vibrating systems*, though, we need to make some sense out of the *angular frequency* ω and the *phase shift* ϕ terms.

C.) Angular Frequency (ω):

1.) Look at the POSITION VERSUS TIME graph of a vibrating body (Figure 9.6). How can I tell you where the body is in its motion at a given point in time? How, for instance, can I tell you that the body is at, say, *Point A* in Figure 9.6?

There are three ways to do the deed. Each is useful in its own way; each is listed below:

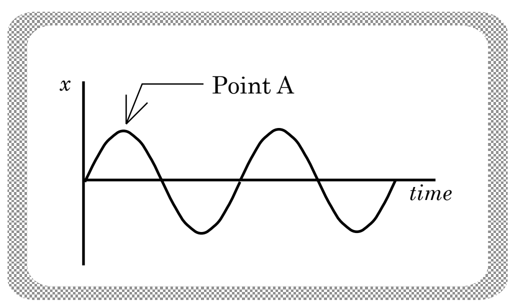


FIGURE 9.6

a.) The first way has already been discussed. I could simply say, "The body is at $x = A$." In that case, I am giving you the "linear displacement" of the body at the point-in-time of interest.

Though simple, it is not very useful if we want to know something about *how fast* the oscillations are taking place. That is, if I know the *time* it takes for the body to get from $x = 0$ to $x = A$, dividing the time into the *displacement* will give only the <u>*average*</u> *velocity* over the motion--a none-too-useful commodity in most cases.

b.) Another possibility is to say, "The body is one quarter of a cycle through its motion." In that case, I am giving you the *cyclic displacement* of the body.

If I additionally tell you how long it takes to achieve that position within its cyclic motion, we can divide the time into that cyclic displacement and come up with an expression for the body's *frequency* in *cycles per second.*

For oscillatory motion, *frequency* measurements are very useful.

c.) Another more exotic possibility is to say, "The body is *$\pi/2$ radians* through its motion." In this case, I would be giving you the angular displacement of the body.

This very peculiar way of characterizing the position of a vibrating body is made simply by looking at the graph of a sine function (see Figure 9.7). Notice that a body having completed one full cycle has moved through an angular displacement of *2π radians*. Following logically, a body having moved through one-half cycle has displaced an angular measure of *π radians* and a body having moved through one-quarter cycle has displaced *$\pi/2$ radians*. In other words, if you understand the language, an angular displacement can tell you where a body is in its motion just as well as a cyclic displacement can.

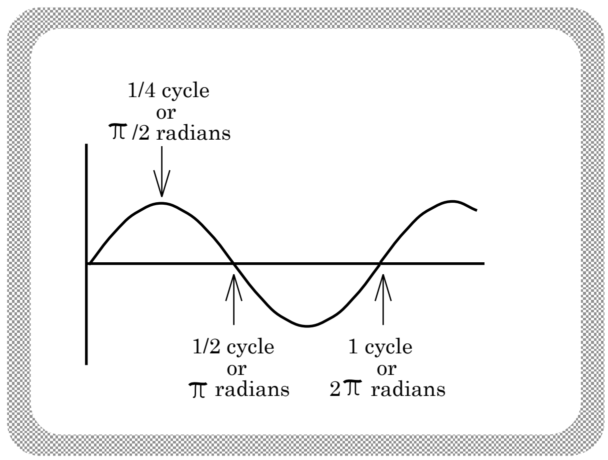


FIGURE 9.7

If, further, we divide this radian-displacement by the time required to get to that position, we end up with an expression for the body's angular frequency ω in *radians per second.*

2.) *One cycle* is the equivalent of an angular measure of *2π radians*. That means that oscillatory motion whose frequency is *1 cycle/second* has an angular frequency of *2π radians/second*. Expanding this, it becomes obvious that the relationship between frequency ν and angular frequency ω is:

$$\omega = 2\pi\nu.$$

3.) Reconsidering $x(t) = A \sin(\omega t + \phi)$, the angular frequency ω governs *how fast the function, hence body position, changes.* Large ω means it takes very little time for ωt to increment by 2π (i.e., move through one cycle), which means the *period* of the function and the body's motion is small. This corresponds to a *high frequency* oscillation. A small ω does just the opposite.

D.) Phase Shift (ϕ):

1.) A typical sine wave function is characterized by the graph shown in Figure 9.8 and is mathematically written as:

$$x(t) = A \sin(\omega t).$$

a.) This expression predicts that at $t = 0$, $x = 0$ (i.e., put in $t = 0$ and you get $x = 0$!). What's more:

b.) *Just after* $t = 0$, the value of $A \sin(\omega t)$ is *positive* and gets larger as time proceeds, just as the graph shows.

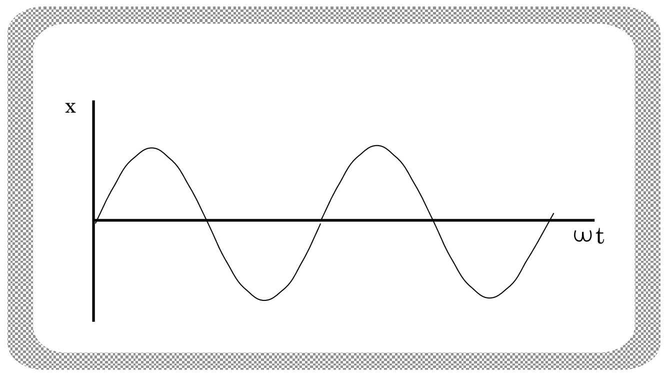


FIGURE 9.8

BIG NOTE: The displacement variable $x(t)$ is measured from EQUILIBRIUM. That means that if you look at a graph of $x(t)$ and see that that variable is getting larger over a particular time interval (either large in a positive sense or larger in a negative sense), it means that the function is modeling motion that is moving AWAY FROM equilibrium.

c.) The problem arises when we do not want the body to be at equilibrium ($x = 0$) at $t = 0$. For instance, what do we do if we want it to be at $x = A$ when we start the clock (i.e., at $t = 0$)? Dealing with such problems is exactly what the phase shift ϕ is designed to do. It allows one to make compensations in the math so that a sine function can be used to characterize oscillatory motion that doesn't assume $x = 0$ at $t = 0$.

2.) Easy Example: Assume we define the position of an oscillating body as $x = +A$ at $t = 0$. How can we use a *sine function* to characterize that motion?

a.) Notice that if we shift the vertical axis of the *sine wave* shown in Figure 9.9a (next page) by $\pi/2$ *radians* (see Figure 9.9b for the shifted version), we end up with a graph that gives us $x = +A$ at $t = 0$ (Figure 9.9c). In other words, adding $\pi/2$ to the *sine's angle* will do for us exactly what we want.

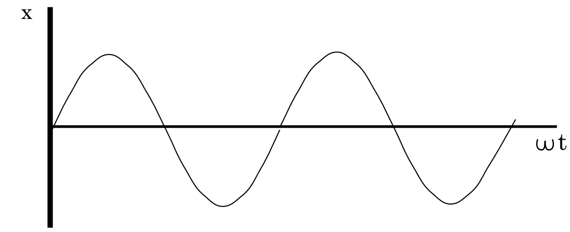


FIGURE 9.9a

b.) Mathematically, we are suggesting that the correct function is:

$$x(t) = A \sin(\omega t + \pi/2).$$

i.) To check, we know what the function's value should be at $t = 0$: it should be $x = +A$.

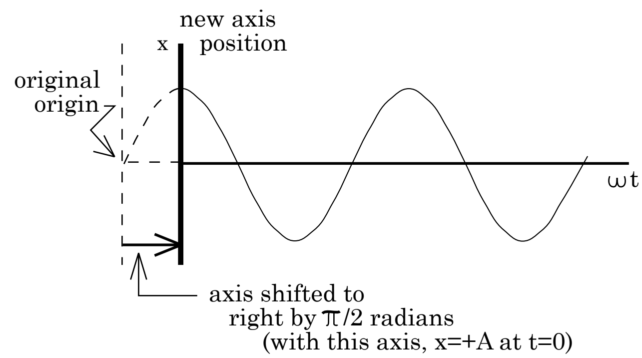


FIGURE 9.9b

ii.) Plugging $t = 0$ into the function we are testing yields:

$$\begin{aligned} x(t=0) &= A \sin(\omega(0) + \pi/2) \\ &= A \sin(\pi/2) \\ &= A. \end{aligned}$$

Our function works!

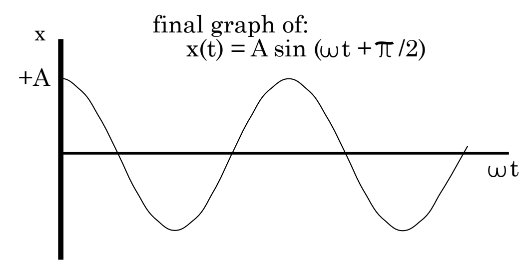


FIGURE 9.9c

c.) The moral: The *phase shift* tells us how much we have to translate (shift) the vertical axis to define the correct displacement x at $t = 0$.

Note: A "+" *phase shift* shifts the axis to the right; a "-" *phase shift* shifts the axis to the left.

3.) In the case above, it was obvious that the shift needed to be $\pi/2$ *radians*. Unfortunately, not all problems are this easy. How does one determine the *phase shift* for more complex situations?

a.) Assume you know where an oscillating body is supposed to be at $t = 0$. The key to determining the general *sine wave function* that will fit the situation lies in evaluating the displacement equation $x(t)$ at a known point in time (preferably at $t = 0$), then solving that expression for the appropriate ϕ. The process will be formally presented using the relatively easy case cited above. We will then try the approach on more difficult problems.

4.) Advanced Example #1: Assume that at $t = 0$, $x = +A$.

a.) Putting that information into our displacement expression

$$x(t) = A \sin\left(\omega t + \phi\right)$$

yields

$$A = A \sin(\omega(0) + \phi).$$

b.) Dividing by A and multiplying w by zero gives us:

$$\begin{aligned} 1 &= \sin(\phi) \\ \Rightarrow \quad \phi &= \sin^{-1}(1) \\ &= 1.57 \qquad \text{(i.e., } \pi/2\text{).} \end{aligned}$$

This is exactly what we expected.

c.) Knowing ϕ for one point in time means we know it for all points in time (ϕ is a constant for the motion). Putting it back into our general algebraic expression for the *displacement* gives us:

$$x(t) = A \sin(\omega t + 1.57).$$

Note: In most problems, you will have already determined both A and w. That is, both will have numeric values. As an example, if $A = 2$ *meters* and $\omega = 7.5$ *radians/second*, the finished expression will look like:

$$x(t) = 2 \sin(7.5t + 1.57).$$

5.) Example #2: Determine the general algebraic expression for the *displacement* of a spring-mass system whose *position* at $t = 0$ is $(3/4)A$ going *away from equilibrium* (see Figure 9.10a and 9.10b on the next page).

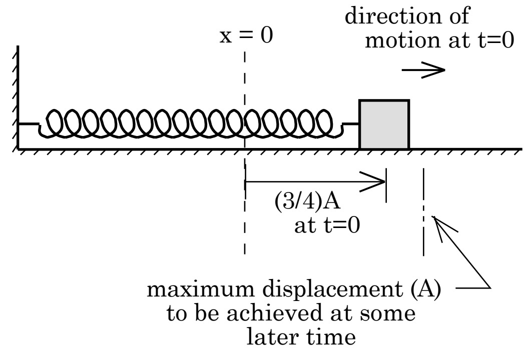


FIGURE 9.10a

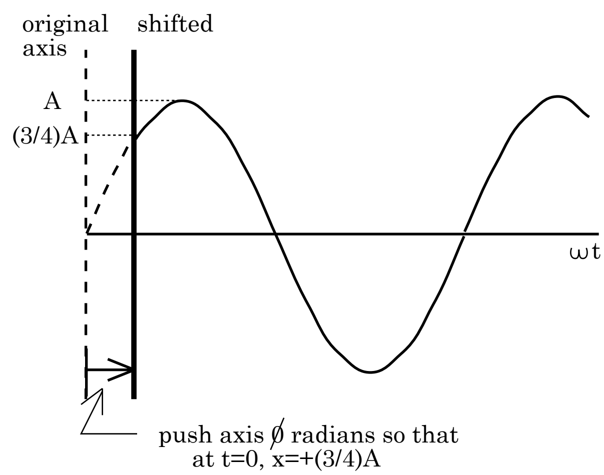


FIGURE 9.10b

a.) In general:

$$x(t) = A \sin\left(\omega t + \phi_1\right).$$

b.) Substituting $t = 0$ and $x = (3/4)A$ into our general equation yields:

$$(3/4)A = A \sin\left(\omega(0) + \phi_1\right).$$

c.) Dividing by A and multiplying ω by zero gives us:

$$3/4 = \sin(\phi_1),$$

which implies that ϕ_1 is the angle whose *sine* is *3/4*, or

$$\phi_1 = \sin^{-1}(3/4)$$
$$= .848 \text{ radians}.$$

d.) Putting our value for ϕ back into our general algebraic expression for the *displacement* gives us:

$$x(t) = A \sin(\omega t + .848).$$

e.) By shifting the axis of the sine wave by *.848 radians* (see Figure 9.10b), we get a graph that has the body's position equal to *.75A* at *t = 0* and that additionally has the displacement proceeding *away from equilibrium* just after *t = 0*.

6.) Example #3--a little different twist: Determine the general algebraic expression for the displacement of a spring/mass system whose position at $t = 0$ is *(3/4)A going <u>toward</u> equilibrium*.

Note: This is almost exactly the same as the problem in #5. The only difference is in the direction of the motion just after $t = 0$. Proceeding through the steps:

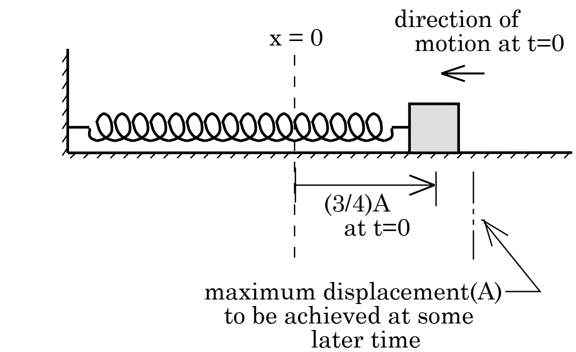


FIGURE 9.11a

a.) In general:

$$x(t) = A \sin\left(\omega t + \phi_2\right).$$

b.) Substituting $t = 0$ and $x = (3/4)A$ into our general equation yields:

$$(3/4)A = A \sin\left(\omega(0) + \phi_2\right).$$

c.) Dividing by A and multiplying ω by zero gives us:

$$3/4 = \sin(\phi_2),$$

which implies that ϕ_2 is the angle whose *sine* is $3/4$, or

$$\phi_2 = \sin^{-1}(3/4)$$
$$= .848 \text{ radians.}$$

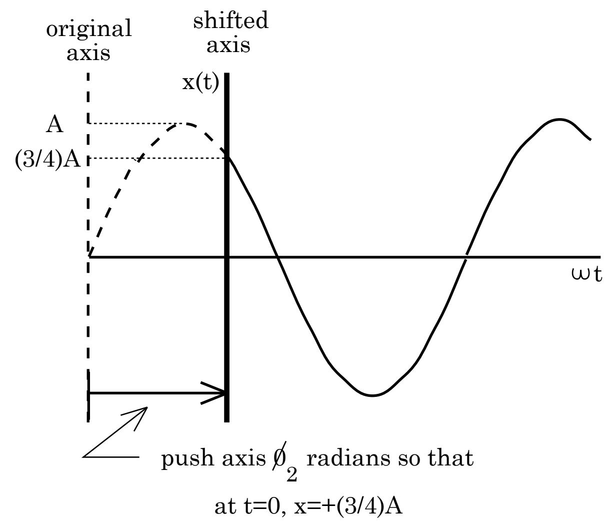


FIGURE 9.11b

d.) THE SNAG: This suggests that the angle ϕ_2 equals the angle ϕ_1, which clearly can't be the case (see Figure 9.12). What we need is an angle that predicts motion that proceeds *back toward equilibrium* just after $t = 0$ . . . not an angle that predicts motion that proceeds *away from* equilibrium after $t = 0$.

Put a little differently, it is clear from the sketch that there are *two* phase shifts that can put $x = (3/4)A$

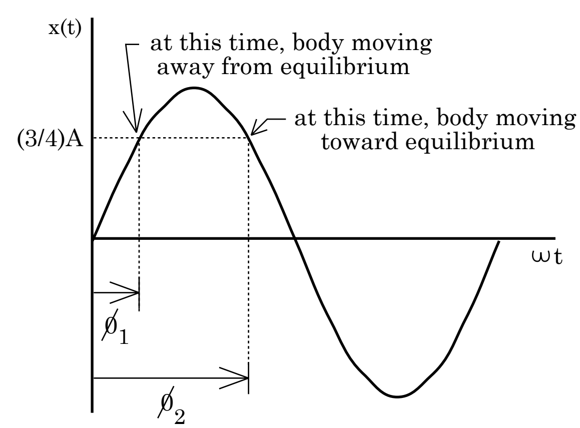


FIGURE 9.12

at $t = 0$. The first (i.e., ϕ_1) is the one we used in #5. It corresponds to the situation when, just after $t = 0$, the motion proceeds *away from equilibrium* (look at the graph--the value for x gets *more* positive as time progresses).

The second phase shift (ϕ_2) is the one we want here. It makes $x = (3/4)A$ at $t = 0$, and it also has the displacement going *back toward* equilibrium as time progresses.

e.) To determine ϕ_2, we need to use the symmetry of the *sine function* (see Figure 9.13). Notice from the figure that the phase shift (ϕ_2) is equal to π - *.848 radians*, or *2.29 radians*.

Using this, the final expression becomes:

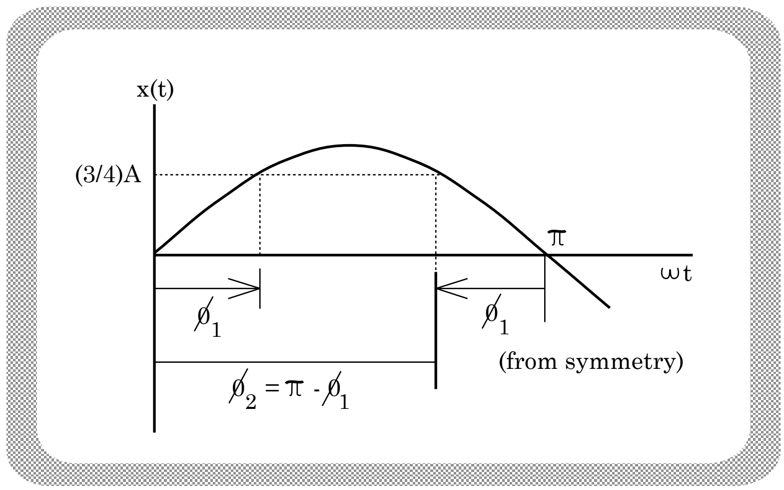


FIGURE 9.13

$$x(t) = A \sin(\omega t + 2.29).$$

f.) Bottom line: Before deciding if the angle your calculator produces is correct, *make a sketch of a sine wave and decide whether you need ϕ_1 or ϕ_2.*

7.) Example #4: Determine a general algebraic expression for the displacement of an oscillating body whose position at $t = 0$ is *(-3/4)A* going *away from* equilibrium (see Figures 9.14a). Assume also that $A = .6$ *meters* and $\omega = 12$ *rad/sec.*

a.) Using the same approach as before:

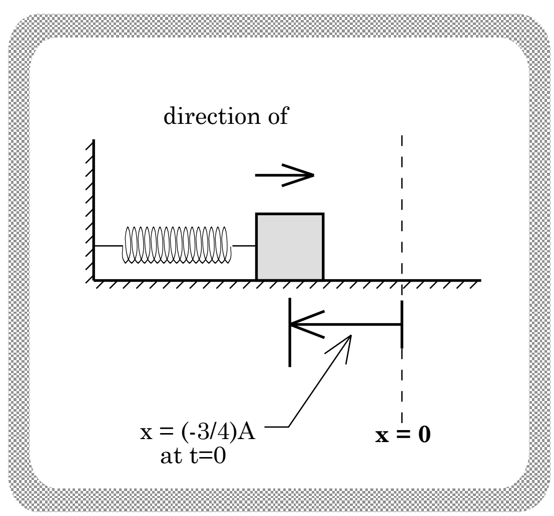


FIGURE 9.14a

$$x(t) = A \sin\left(\omega t + \phi_3\right)$$

$$\Rightarrow \quad (-3/4)(.6) = (.6) \sin\left(\omega(0) + \phi_3\right)$$

$$\Rightarrow \quad -3/4 = \sin(\phi_3)$$

$$\Rightarrow \quad \phi_3 = \sin^{-1}(-3/4)$$

$$= -.848 \text{ radians.}$$

b.) A <u>negative</u> *phase shift* moves the axis to the *left*. Again, there are *two* positions where an axis can be placed so that at *t = 0, x = (-3/4)A* (see Figure 9.14b). The first, corresponding to an *angular shift* of the axis of ϕ_3, has the body moving *toward equilibrium* just after *t = 0*; the second, corresponding to an angular shift of the axis of ϕ_4, has the body moving *away from* equilibrium just after *t = 0*.

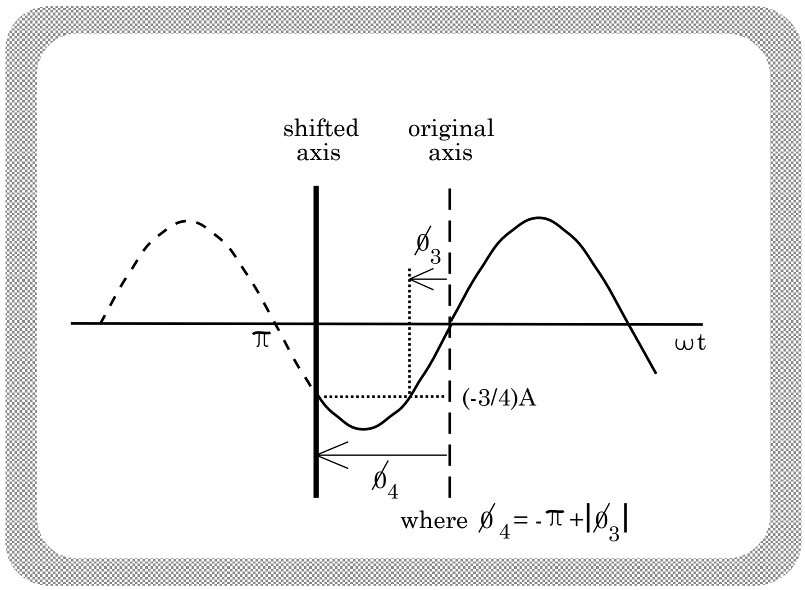


FIGURE 9.14b

In our example, the appropriate angular shift is ϕ_4. The displacement expression is, therefore:

$$x(t) = (.6) \sin\left(12t + 2.29\right).$$

8.) The technique for determining *phase shifts* is simple. Put the *t = 0* value for displacement into $x(t) = A \sin(\omega t + \phi)$, solve algebraically for ϕ, and your calculator will crank out a number for you.

a.) If the calculator's number is *positive*, shift the axis to the right. If the number is *negative*, shift the axis to the left.

b.) The only thing tricky about the operation: in almost all cases there will be two possible axes (i.e., shift angles) that will correspond to the required *t = 0* displacement. Determine which is appropriate by noting whether the motion is proceeding *away from* equilibrium or *toward* equilibrium. That information will dictate whether you can use your calculator-provided *phase shift* value or whether you will have to add or subtract π.

c.) Whatever the case, you should end up with an expression that looks something like $x(t) = 2 \sin (7.5t + 1.57)$.

E.) Energy in a Vibrating System:

1.) Consider the motion of a mass attached to a vibrating spring:

a.) At the extremes, the body's *velocity* is zero (it's at a turn-around point), its *position* is a maximum (i.e., $x = A$), and all the *energy* in the system is *potential energy*.

That is, at the extremes:

$$E_{total} = U\,(x_{max}).$$

b.) The *potential energy* function for a spring system is $(1/2)kx^2$. This means:

$$\begin{aligned} E_{total} &= U\,(x_{max}) \\ &= (1/2)kx_{max}^{\ 2} \\ &= (1/2)kA^2. \end{aligned}$$

c.) *Assuming there is no energy loss* during the motion, the *amplitude* of the motion remains constant and the total energy of the system is conserved. The energy flows back and forth between being *potential* and *kinetic*, but the sum of the two is always equal to $(1/2)kA^2$.

F.) A summary example:

1.) You have a spring hanging from the ceiling. You know that if you elongate the spring by 3 meters, it will take 330 nts of force to hold it at that elongated position.

The spring is hung and a 5 kg mass is attached. The system is allowed to reach equilibrium; then is displaced an additional 1.5 meters and released. For this system, what is the:

a.) *Spring constant*?

b.) *Angular frequency*?

c.) *Amplitude*?

d.) *Frequency*?

e.) *Period*?

f.) *Total energy*?

g.) *Maximum velocity* of the mass?

h.) *Position* of the mass at maximum velocity?

i.) *Maximum acceleration* of the mass?

j.) *Position* of the mass at maximum acceleration?

k.) General *algebraic expression* for the position of the mass as a function of time, assuming that at $t = 0$ the body's position is located at $y = -A/4$ going away from equilibrium?

2.) Solutions:

a.) F/x = 110 nt/m; **b.)** $(k/m)^{1/2}$ = 4.7 rad/sec; **c.)** 1.5 m (from observation); **d.)** $\omega / 2\pi$ = .75 hz; **e.)** $1/\nu$ = 1.33 sec/cycle; **f.)** $(1/2)kA^2$ = 123.75 joules; **g.)** ωA = 7.05 m/s; **h.)** at equilibrium position; **i.)** $\omega^2 A$ = 33.135 m/s^2; **j.)** at the extremes; **k.)** either x(t) = 1.5 sin (4.7t + 3.39) or x(t) = 1.5 sin (4.7t - 2.89).

G.) Another Kind of Vibratory Motion--The Pendulum:

1.) Consider a swinging pendulum bob of mass m at the end of a string of length L positioned at an arbitrary angle θ as shown in Figure 9.15. What is the system's *frequency*, *period* of oscillation, *angular frequency*, etc.?

a.) We will begin the same way we did with the spring. If the Newton's Second Law equation for this situation matches the form:

acc. + (constant) disp. = 0,

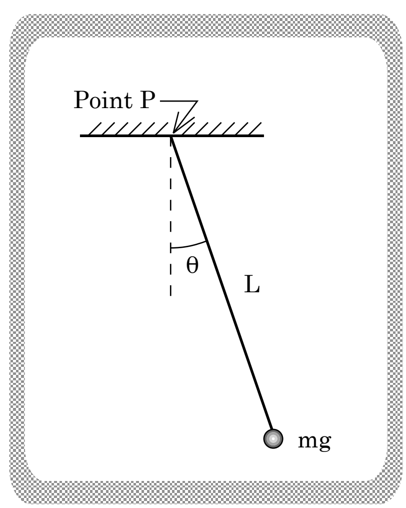


FIGURE 9.15

we know the motion will be *simple harmonic* and we know that the *constant* will numerically equal the *angular frequency squared*.

b.) The only difference between this situation and the spring situation is that in this case, the pendulum bob is moving in a *rotational* sense around the string's point of attachment *P*. The version of N.S.L. that is applicable here, therefore, is the rotational version.

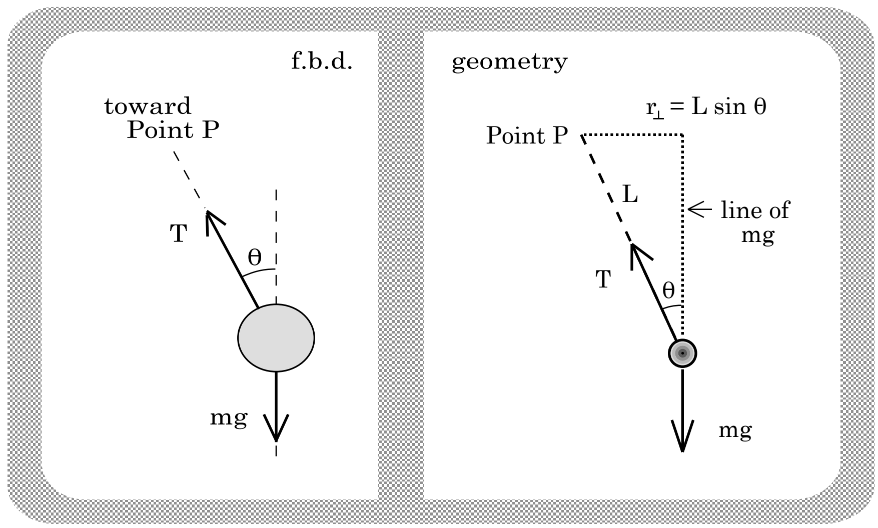


FIGURE 9.16a FIGURE 9.16b

c.) Figure 9.16a shows the *free body diagram* for the set-up. Figure 9.16b shows that the *torque* about *Point P* due to the tension *T* is zero (the tension force passes through *Point P*), and the *torque* due to gravity is $mg\,(L \sin\,\theta)$ (in this case, $r_\perp$ is $L \sin\,\theta$). Remembering that the *moment of inertia* for a point mass is $I_{ptmass} = mL^2$, the rotational counterpart to Newton's Second Law yields:

$\underline{\sum \Gamma_p}$:

$$-mg\,(L \sin\,\theta) = I\alpha$$
$$= (mL^2)\,a$$

which implies:

$$\alpha + (g/L) \sin\,\theta = 0.$$

d.) This is not the form for which we were hoping. Fortunately, if θ is <u>small</u> and measured in *radians*, $\sin\,\theta = \theta$ (put your calculator in *radian* mode and see what *sin (.02)* is--you should find that it is .01999999--.02 to a good approximation).

e.) Making the *small angle approximation*, we get:

for θ <<:

$$\alpha + (g/L)\ \theta = 0.$$

f.) Running a parallel from our spring experience, we know that the oscillation's *angular frequency* must be:

$$\omega = (g/L)^{1/2}.$$

g.) With the angular frequency ω, we can determine general algebraic expressions for the motion's frequency ($\omega/2\pi$) and period ($1/\nu$).

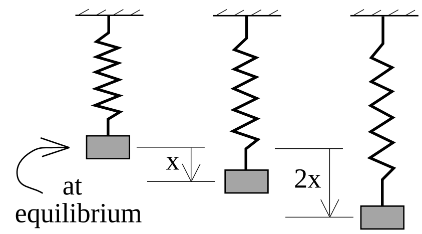
2.) Reiteration: If you are ever asked to determine either the *period* or *frequency* of an exotic oscillatory system, use N.S.L. and see if you can put the resulting *equation of motion* into the form:

acc. + (constant)(displ.) = 0.

If you can do so, the motion will be simple harmonic in nature and the *angular frequency* will equal *the square root of the constant*. From there you can easily determine the motion's *frequency* and/or *period*.

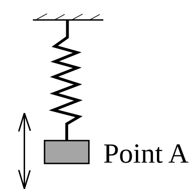
QUESTIONS

9.1) An ideal spring attached to a mass m = .3 kg provides a force equal to -kx, where k = 47.33 nt/m is the spring's spring constant and x denotes the spring's displacement from its equilibrium position. Let's assume that when such a spring is displaced a distance x = 1 meter, the period of oscillation (this is defined as the amount of time required for the system to oscillate through one complete cycle) is T = .5 seconds per cycle.



- **a.)** When the mass is displaced a distance 2x = 2 meters, what is its new period?
- **b.)** Given the numbers in the original statement of the set-up, would it have been possible for the period to have been any other number other than .5 seconds per cycle? Explain.

9.2) A vertical spring/mass system oscillates up and down. At $t = 0$, the mass is at *Point A* moving downward. Through how many cycles will the system have moved by the time the mass has passed by *Point A* five times, not including its first passage at $t = 0$?



9.3) When you attach a mass to an ideal spring, the force *F* provided to the mass by the spring will be proportional to the displacement *x* of the mass/spring system from its equilibrium position. Algebraically, that proportionality can be written as an equality equal to $F = -kx$, where *k* is the proportionality constant and is called *the spring constant*. One of the things that is interesting about the oscillatory motion of the mass attached to an ideal spring is that the mass's motion will have a single period *T*. That is, it will always take the same amount of time for the mass to oscillate through one cycle, no matter what the initial displacement was. Having said that:

- **a.)** Sketch the *Force versus Displacement* graph for an ideal spring. Remember that the displacement of a spring from its equilibrium position can be either positive or negative.
- **b.)** Briefly, explain why the period of an ideal spring/mass system doesn't change if the initial displacement of the mass is increased or decreased.
- **c.)** Now for the fun part. Consider a second, *non-ideal* spring whose force expression is $-bx^3$, where *b* is some spring constant. On the graph you produced in *Part a*, make an approximate sketch of the *Force versus*

Displacement graph for this spring force (don't get anal about this--you don't need numbers, just show the trend of the force as x goes positive and negative).

d.) Attach the non-ideal spring to the same mass you used in *Part a* and *b*. It is possible to displace this spring/mass system so that when released, it oscillates with the same period as was the case with the ideal spring used in *Part a*. Take that displacement, double it, displace the system that doubled distance, and release. Will the period of the resulting oscillation be greater than, less than, or the same as T?

9.4) Can a spring have a force function of $-kx^4$? Explain.

9.5) You have access to Geppetto's Workshop, complete with Newton scales, meter sticks, balances--all sorts of science-y things. Someone gives you an ideal spring and asks you to determine its spring constant. How might you do that?

9.6) Most people know that frequency measures the number of cycles through which an object oscillates per unit time. What does *angular frequency* measure?

9.7) A fixed length of string is cut and loops are made at both ends. The upper end-loop is placed over a ceiling hook while the lower end-loop is used to support a hook-mass m. The mass is pulled to the side and released making a pendulum that swings back and forth. The period is measured as T. The original mass is removed and a second hook-mass from the same mass set, this one of mass $10m$, is placed on the string and made to swing back and forth with the same amplitude. The new period is found to be larger than T.

a.) Does this mean the pendulum is swinging faster or slower?

b.) Some students look at the data and conclude that the pendulum's period is a function of the bob's mass. In fact, this isn't true! What is probably causing the disparity in the periods?

9.8) Newton's Second Law is used to sum up the forces acting on an oscillating mass. The resulting expression is then manipulated and found to have the form $(d^2x/dt^2) + bx = 0$. Having access to this expression:

a.) What can you say about the system's angular frequency of the system?

b.) What can you say about the system's frequency?

c.) What can you say about the system's period?

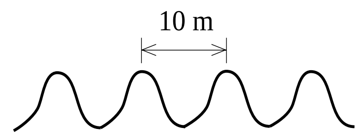
9.9) What is the single characteristic that is common to all vibrating (oscillatory) systems?

9.10) The acceleration of gravity on earth is approximately six times that of the acceleration on the moon. A pendulum on earth has a period of *1 second per cycle*. Will the pendulum's period change if it is used on the moon? If so, how so?

9.11) Double the length of a pendulum arm. How will the pendulum's frequency change? How will the pendulum's period change?

9.12) How are frequency and period related?

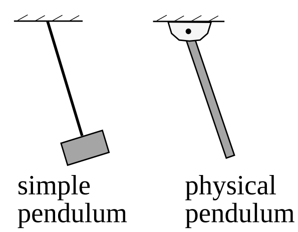
9.13) You are sitting on a jetty. You notice ocean waves are coming in approximately 10 meters apart. It takes 30 seconds for *three crests* to pass you by. What is the frequency, period, and angular frequency of the wave train?



9.14) A spring with spring constant $k = .25$ *newtons per meter* vibrates with frequency $\nu = .5$ *hertz*. Across the lab, a string with a small mass $m = .15$ *kg* attached to it is made into a simple pendulum.

- **a.)** If the frequency of the pendulum and the frequency of the spring are to be the same, approximately how long must the string be?
- **b.)** Why are you being asked for an approximate answer? That is, given what you know, why can't you give an exact answer?
- **c.)** For the frequency to be good, is there any limit on the size of the oscillations of the pendulum?

9.15) What is the difference between a simple pendulum and a physical pendulum of same mass and length? What approach would you use to derive from scratch an expression for the period of either?



9.16) You live in California (Los Angeles). You're a physics teacher, complete with sadistic streak. You have your students calculate the period of a pendulum system. They determine that value to be T. You then claim that no matter how good and precise your students' set-up is, its period will never exactly equal the theoretically calculated value, *even if your students do the experiment in a vacuum*. What are you talking about? (Note: This isn't obvious--think about the parameters that determine a pendulum's period, and how they might be off). Once you've figured out the problem, approximate by how much your theoretical period will be off (note that the latitude of LA is approximately 22^o). Is this going to be noticeable?

9.17) Consider the expression $x = A \sin(\omega t + \delta)$.

a.) What does the A term do for you ?
b.) What does the ω term do for you?
c.) What does the δ term do for you?
d.) What does the expression in general do for you?

9.18) Identify a system in which a restoring force exists, and identify what the restoring force actually *is* in the system.

9.19) Identify a system in which a restoring torque exists, and identify what force provides that restoring torque.

PROBLEMS

9.20) A spring/mass set-up oscillating in the vertical is found to vibrate with an amplitude of .5 meters and a period of .3 seconds per cycle. If the mass is 1.2 kg, determine:

a.) The frequency of oscillation;
b.) The angular frequency;
c.) The spring constant;
d.) The maximum velocity (in general, where does this happen);
e.) The maximum acceleration (in general, where does this happen);
f.) How much energy is wrapped up in the system?

9.21) A .25 kg mass sliding over a frictionless horizontal surface is attached to a spring whose spring constant is 500 nt/m. If the spring's maximum velocity is 3 m/s, determine the motion's:

a.) Angular frequency;
b.) Frequency;
c.) Period;
d.) Amplitude;
e.) Total energy;
f.) Maximum force applied to the mass.

9.22) A body's motion is characterized by the expression:

$$x(t) = .7 \sin (14t - .35).$$

Determine the motion's:

a.) Amplitude?
b.) Angular frequency?
c.) Frequency?
d.) Position at t = 3 seconds?
e.) Position at t = 3.4 seconds?
f.) Velocity at t = 0?
g.) Acceleration at t = 0?

9.23) A pendulum consists of a small, 2 kg weight attached to a light string of length 1.75 meters. The pendulum is set up on a distant planet and set in motion. Doing so, it is observed that its period is 2 seconds per cycle. What is the acceleration of gravity on the planet?

9.24) The Newton's Second Law equation shown below came from the analysis of an exotic pendulum system oscillating with a small angular displacement. It is:

$$a + (12g/7L)\ \theta = 0$$

a.) Given the information provided above, how can you tell that the system oscillates with *simple harmonic motion*?
b.) What is the system's theoretical *frequency of oscillation* if the pendulum length is assumed to be 1.3 meters?

9.25) A 3 kg block is attached to a vertical spring. The spring and mass are allowed to gently elongate until they reach equilibrium a distance .7 meters below their initial position. Once at equilibrium, the system is displaced an additional .4 meters. A stopwatch is then used to track the position of the mass as a function of time. The clock is started when the mass is at *y = -.15 meters* (relative to equilibrium) moving *away from* equilibrium. Knowing all this, what is:

a.) The *spring constant*?
b.) The oscillation's *angular frequency*?
c.) The oscillation's *amplitude*?

d.) The oscillation's *frequency*?
e.) The *period*?
f.) The *energy* of the system?
g.) The *maximum velocity* of the mass?
h.) The *position* when at the maximum velocity?
i.) The *maximum acceleration* of the mass?
j.) The *position* when at the maximum acceleration?
k.) A general *algebraic expression* for the position of the mass as a function of time?

9.26) A tunnel is dug through the earth from the North Pole to the South Pole. When done, Jack (the idiot) goes for the thrill of his life and jumps into the hole. The gravitational force on him is always directed toward the earth's center, so Jack ends up oscillating back and forth between the two poles.

In the chapter on Gravitation, we derived an expression for the magnitude of the gravitational force acting on a mass a distance r units from the earth's center, where $r < r_e$ with r_e being the earth's radius.

Tailored to our situation, that expression is:

$$F_J = -\left[\frac{Gm_e m_J}{r_e^{\,3}}\right]r$$

where m_e and r_e are the mass and radius of the earth, respectively, m_J is Jack's mass, and r is Jack's position along the *y-axis* (we are assuming the tunnel is in the vertical).

Jack's father misses him. As such, Papa has hired a surveillance satellite whose orbit is such that every time Jack's *head* emerges momentarily from the hole, the satellite and its cameras are directly above to snap photos.

For this to work, what must the satellite's period be?